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Conservation laws of the BBM equation

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Abstract. It is proved that the BBM equation $u_{xxt} = u_t - uu_x$ has no conservation laws except those found by Benjamin, Bona and Mahony.

1. Introduction

It is well known that the Korteweg–de Vries equation,

$$u_t + u_x + uu_x + u_{xxx} = 0,$$

has N-soliton solutions which describe elastic interaction of solitary waves in shallow water. This remarkable property is related to the fact that the KdV equation is a completely integrable Hamiltonian system without stochastisation possessing an infinite number of conservation laws.

As an alternative model for the long wave motion in nonlinear dispersive systems, Benjamin *et al* (1972) proposed the regularised long wave (RLW, or BBM) equation,

$$u_t + u_x + uu_x - u_{xxt} = 0.$$

These authors (see also Bona *et al* 1983) argue that both equations are valid at the same level of approximation, but that the BBM does have some advantages over the KdV from the computational mathematics viewpoint.

Numerical experiments first carried out by Abdulloev *et al* (1976) and then by others (see Bona *et al* 1983) show that the BBM equation admits soliton solutions whose interaction is inelastic though close to elastic. Considering the BBM equation as a 'deformation' of the KdV equation, we see that the latter displays surprising stability of its seemingly fragile mathematical properties. Therefore a natural question arises as to whether the behaviour of the solutions of the BBM equation can be explained in terms of conservation laws.

Below, we write the BBM equation as $u_{xxt} = u_t - uu_x$ (it takes this form after replacing u by $-1-u$ in the original version). Olver (1979) showed that this equation has no other conserved densities depending only on x, u, u_x, u_{xx}, ... than those indicated by Benjamin *et al* (1972): u (mass), $(u^2+u_x^2)/2$ (energy), and $u^3/3$ (momentum). This result, however, does not imply that the BBM equation is not a completely integrable Hamiltonian system, since there might exist other conserved densities which depend also on t and t-derivatives of u and u_x. Note that from the point of view adopted by Olver these conserved densities, if they exist, can be considered also as functions of

x, u and x-derivatives of u which are however non-local, since, for example, $u_t = (1 - D_x^2)^{-1}(uu_x)$.

In this paper we prove that the BBM equation does not possess further conservation laws of this kind, either. Perhaps this result accounts for the inelastic mode of interaction of the soliton solutions. However, the numerical works referred to above seem to suggest that the BBM equation may have other conserved densities which must be really non-local (i.e. not functions of t, x, u and arbitrary derivatives of u) because of our result.

The method we apply for enumerating the conservation laws is based on the calculation of a certain part ($E_1^{1,1}$) of the Vinogradov's spectral sequence (Vinogradov 1978) associated with the given equation, which is carried out by solving an auxiliary linear differential equation. Our method is valid for a very broad class of (systems of) nonlinear partial differential equations.

In this paper, we present our calculation of the space of conservation laws of the BBM equation together with a brief account of the necessary theory, which we hope will clarify the general character of our method, although all calculations could have been done without referring to the Vinogradov's spectral sequence.

2. Equations

Let $\pi: N \to M$ be a smooth bundle given in local coordinates by $\pi(x_1, \ldots, x_n, u^1, \ldots, u^m) = (x_1, \ldots, x_n)$, let $\pi_k: N_k \to M$ be its kth jet bundle, N_k being the manifold with coordinates x_i, u^j and all partial derivatives of the latter over the former up to the order k, and $R \subset N_k$ a kth-order differential equation with independent variables x_i ranging over M and dependent variables u^j ranging over the fibre of π. Locally, R can be defined as $F_1 = \ldots = F_r = 0$ where F_i are smooth functions on N_k. Globally, R is the set of all points in N_k satisfying $F = 0$ where F is a section of a bundle over N_k with local components $F_1, \ldots, F_r$.

Denote by $R_\infty \subset N_\infty$ the infinite prolongation of R. In the standard coordinate system x_i, u_σ^j with $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ and σ being an arbitrary multi-index consisting of $x_1, \ldots, x_n$, R_∞ is the submanifold obtained by equating F and its various total derivatives to zero.

Example. Take $N = \boldsymbol{R}^3$ with coordinates (x, t, u), $M = \boldsymbol{R}^2$ with coordinates (x, t) and $\pi: M \to N$ given by $(x, t, u) \mapsto (x, t)$. The BBM equation is the submanifold of N_3 given by

$$u_{xxt} = u_t - uu_x. \tag{1}$$

Taking into account all differential consequences of (1)

$$u_{xxxt} = u_{xt} - uu_{xx} - u_x^2,$$

$$u_{xxtt} = u_{tt} - uu_{xt} - u_x u_t, \qquad \text{etc.},$$

we arrive at the submanifold R_∞ of N_∞ which, in this case, is topologically trivial, i.e., admits a global coordinate system. As such one can take

$$x, t, \qquad \{u_k; k \geqslant 0\}, \{v_k; k \geqslant 1\}, \{w_k; k \geqslant 2\} \tag{2}$$

where $u_k = u_{xx\ldots x}$ (k times x), $v_k = u_{tt\ldots t}$ (k times t), $w_k = u_{xt\ldots t}$ ($k-1$ times t).

Let A and $\bar{A}$ stand for the algebras of smooth functions on N_∞ and R_∞, respectively: $\bar{A}=A/\mathscr{I}$ where $\mathscr{I}$ is the ideal of functions vanishing on R_∞. For the BBM equation, $\mathscr{I}$ is differentially generated by the function $F=u_{xxt}-u_t+uu_x$, and $\bar{A}$ is identical with the algebra of all smooth functions of a finite number of variables (2). Obviously, $\bar{A}=\bigcup_{n\geq 0}\bar{A}_n$, $\bar{A}_n$ being the subalgebra of functions depending only on x, t, u_k, v_k, w_k with $k\leq n$.

Here and below, the bar over symbols means restriction of objects and maps to the manifold R_∞. Since the ideal $\mathscr{I}$ is closed under the total derivative operators, the latter can be restricted to R_∞.

Example. In the coordinate system (2) of the BBM equation, the total derivatives over x and t are written as

$$\bar{D}_x=\partial/\partial x+\sum_{k\geq 0}u_{k+1}\,\partial/\partial u_k+\sum_{k\geq 1}w_{k+1}\,\partial/\partial v_k+\sum_{k\geq 2}\left(v_{k-1}-\sum_{i=0}^{k-2}\binom{k-2}{i}v_i\,w_{k-1-i}\right)\partial/\partial w_k$$

$$\bar{D}_t=\partial/\partial t+\sum_{k\geq 0}(\bar{D}_tu_k)\partial/\partial u_k+\sum_{k\geq 1}v_{k+1}\,\partial/\partial v_k+\sum_{k\geq 2}w_{k+1}\,\partial/\partial w_k$$

where $\bar{D}_tu_k$ can be determined recursively:

$$\bar{D}_tu_0=v_1,\qquad \bar{D}_tu_1=w_2,\qquad \bar{D}_tu_{k+1}=\bar{D}_tu_{k-1}-\tfrac{1}{2}\sum_{i=0}^{k}\binom{k}{i}u_iu_{k-i},$$

and we put $w_1=u_1$ and $v_0=u_0$ for brevity.

3. Conservation laws

Let (Ω_0, d_0) stand for the 'horizontal de Rham complex' of the manifold N_∞, i.e., $\Omega_0=\Sigma_{k\geq 0}\Omega_0^k$, where Ω_0^k is the space of all differential k-forms locally written as $\omega=\Sigma f_{i_1\ldots i_k}\,\mathrm{d}x_{i_1}\wedge\ldots\wedge\mathrm{d}x_{i_k}$ with $f_{i_1\ldots i_k}\in A$, and $\mathrm{d}_0\omega$ is obtained from $\mathrm{d}\omega$ by substituting each $\mathrm{d}u^j_\sigma$ by $u_{jx_1}\,\mathrm{d}x_1+\ldots+u_{jx_n}\,\mathrm{d}x_n$ (in other words, by imagining that u^j_σ which is an independent coordinate on N_∞ is really a function of $x_1,\ldots,x_n$). Thus

$$d_0\omega=\sum_{i_1,\ldots,i_k}\sum_i D_i(f_{i_1\ldots i_k})\,\mathrm{d}x_i\wedge\mathrm{d}x_{i_1}\wedge\ldots\wedge\mathrm{d}x_{i_k}.$$

Since the ideal $\mathscr{I}$ is closed under the total derivatives D_i, we can restrict d_0 to the prolonged equation R_∞ and thus arrive at the 'horizontal de Rham complex of the equation R_∞',

$$\ldots\to\bar{\Omega}_0^{k-1}\xrightarrow{\bar{d}_0}\bar{\Omega}_0^k\xrightarrow{\bar{d}_0}\bar{\Omega}_0^{k+1}\to\ldots.\tag{3}$$

Following (Tsujishita 1982, Vinogradov 1984a, b), we define the space of conservation laws of the equation R to be

$$H_0^{n-1}(R)=\mathrm{Ker}\,\bar{d}_0|\bar{\Omega}_0^{n-1}/\mathrm{Im}\,\bar{d}_0|\bar{\Omega}_0^{n-2},$$

the $(n-1)$th cohomology group of the complex (3).

This definition coincides in fact with the usual one. We explain this for the case $M=\boldsymbol{R}^2$ with coordinates x, t. A conservation law is represented by a one-form $\omega=T\,\mathrm{d}x+X\,\mathrm{d}t$, $T, X\in\bar{A}$, which is closed in the complex (3), i.e., $\bar{d}_0\omega=0$. The latter is equivalent to $\bar{D}_xX=\bar{D}_tT$, so that T and X are just the classical conserved density and conserved flux. If $T = \bar{D}_x P$ and $X = \bar{D}_t P$ for some function P, the conservation law is regarded as trivial; correspondingly, the one-form $\omega = \bar{d}P$ is exact and defines the zero cohomology class.

Note that our understanding of trivial conservation laws differs somewhat from that adopted by Olver (1979): the latter demands only that $T = \bar{D}_x P$. Thus for instance, the one-form $(u_x^2 - u^2)\,\mathrm{d}t$ is not exact in the horizontal de Rham complex of the equation $u_{xx} = u$, yet the relation $\bar{D}_x(u_x^2 - u^2) = 0$ yields a trivial (according to Olver) conservation law, since its conserved density is zero. The space of conserved densities is in fact a factor space of $\bar{H}^{n-1}(R)$.

Proposition. If for some equation over $\boldsymbol{R}^2$ *with coordinates* x, t *one has* Ker $\bar{D}_x = C^\infty(t)$ *(the set of all functions of* t*), then the linear dependence of conserved densities implies linear dependence of corresponding conservation laws.*

Proof. Let $\bar{D}_t T = \bar{D}_x X$ and $T = \bar{D}_x P$ for some $P \in \bar{A}$. Then $\bar{D}_x(X - \bar{D}_t P) = 0$ whence by the assumption $X - \bar{D}_t P = f \in C^\infty(t)$ and $X = \bar{D}_t(P + F)$ where F is a primitive of the function f. Observe finally that $T = \bar{D}_x(P + F)$.

4. Kernels of total derivations

We have seen, in § 3, that the size of Ker $\bar{D}_x$ is of a certain interest in the study of conservation laws. We now compute the spaces Ker $\bar{D}_x$ and Ker $\bar{D}_t$ for the BBM equation. We note that the lemma 1 below implies, by virtue of the proposition of § 3, that, for the equation in question, both viewpoints on the linear dependence of conservation laws actually coincide. The two lemmas we are going to prove will be essentially used in § 7.

Lemma 1. Ker $\bar{D}_x = C^\infty(t)$.

Proof. Let $\bar{D}_x f = 0$ and $f \in \bar{A}_n$. The coefficient of u_{n+1} in $\bar{D}_x f$, $\partial f/\partial u_n$, must vanish. Hence f does not depend on u_n and the coefficient of u_n, $\partial f/\partial u_{n-1}$, must vanish. Thus we come up to $\partial f/\partial u_1 = 0$. Similarly, $\partial f/\partial v_n = 0$. Now $\bar{D}_x f = (Xf)u_1 + Yf$ where

$$X = \frac{\partial}{\partial u} - \sum_{i=2}^{n} v_{i-2} \frac{\partial}{\partial w_i}$$

$$Y = \frac{\partial}{\partial x} + \sum_{i=2}^{n} w_i \frac{\partial}{\partial v_{i-1}} + \sum_{i=2}^{n} \left(v_{i-1} - \sum_{j=0}^{i-3} \binom{i-2}{j} v_j w_{i-1-j} \right) \frac{\partial}{\partial w_i}.$$

Set $X_1 = [X, Y]$, $X_2 = [X, X_1]$, $X_{i+1} = [X_1, X_i]$, $i \geqslant 2$. Then $X_n = -\partial/\partial v_{n-1}$. Hence $Xf = Yf = 0$ implies $\partial f/\partial v_{n-1} = 0$. The coefficient of v_{n-1} in $\bar{D}_x f$ equals now $\partial f/\partial w_n$. Since this must vanish we have $f \in \bar{A}_{n-1}$, and the lemma is proved by induction.

Lemma 2. Ker $\bar{D}_t = C^\infty(x)$.

Proof. From $f \in$ Ker $\bar{D}_t \cap \bar{A}_n$ we derive step by step $\partial f/\partial v_n = \partial f/\partial w_n = \ldots = \partial f/\partial v_2 = \partial f/\partial v_1 = 0$. Then we have $\bar{D}_t f = (X_0 f)v_1 + (X_1 f)w_2 + Yf$ where

$$X_0 = \frac{\partial}{\partial u} + \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_4} + \ldots$$

$$X_1 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_3} + \frac{\partial}{\partial u_5} + \ldots$$

$$Y = \frac{\partial}{\partial t} + \sum_{k=1}^{\infty} (\bar{D}_t u_{2k} - v_1) \frac{\partial}{\partial u_{2k}} + \sum_{k=1}^{\infty} (\bar{D}_t u_{2k+1} - w_2) \frac{\partial}{\partial u_{2k+1}}.$$

Putting $X_{i+1} = -[X_0, [X_i, Y]]$, $i \geq 1$, we get $X_n = \partial/\partial u_n$ on the algebra of functions $C^\infty(x, t, u, u_1, \ldots, u_n, v_1, w_2)$, which completes the proof of the lemma. Note also that $X_{n-1} = \partial/\partial u_{n-1}$ on this algebra, which will be used in § 7.

5. Vinogradov's spectral sequence

Since the notion of conservation law has an essentially homological nature, it can be studied with the machinery of algebraic topology. The suitable tool is Vinogradov's spectral sequence (Vinogradov 1978, 1984a, Tsujishita 1982). For a general account of spectral sequences, see Godement (1958).

The Vinogradov's spectral sequence of a differential equation $R \subset N_k$ is constructed in the following way. In the de Rham algebra Ω of the manifold N_∞, consider the ideal $\mathscr{C}$ consisting of all differential forms vanishing on the manifolds $j_\infty(s)(M) \subset N_\infty$ where $j_\infty(s): M \to N_\infty$ is the infinite jet of a local section s of the bundle π, namely,

$$\mathscr{C} = \{\omega \in \Omega \,|\, j_\infty(s)^*(\omega) = 0 \;\; \forall\, s \in \Gamma_{\text{loc}}(\pi)\}.$$

That is, ω lies in $\mathscr{C}$ if it vanishes as soon as we imagine the u^j_σ's to be really the partial derivatives of functions of $x_1, \ldots, x_n$.

Example. For the bundle $\boldsymbol{R}^3 \to \boldsymbol{R}^2$ mentioned in § 2 the form $\delta u_\sigma = \mathrm{d}u_\sigma - u_{\sigma x}\,\mathrm{d}x - u_{\sigma t}\,\mathrm{d}t$ belongs to $\mathscr{C}$ since every function $u(x, t)$ satisfies $\mathrm{d}u_\sigma = (\partial u_\sigma/\partial x)\,\mathrm{d}x + (\partial u_\sigma/\partial t)\,\mathrm{d}t$. It is easy to prove that the ideal $\mathscr{C}$ is generated by the set ${}^1\mathscr{C} = \mathscr{C} \cap \Omega^1$ of its elements of degree one and that the one-forms δu_σ (σ being arbitrary multi-index) constitute a basis for the A-module ${}^1\mathscr{C}$, i.e., every element of ${}^1\mathscr{C}$ is a unique linear combination of δu_σ's with function coefficients.

Obviously, $\mathscr{C}$ is closed under the exterior differentiation ($d\mathscr{C} \subset \mathscr{C}$) and so are its powers $\mathscr{C}^p = \mathscr{C} \wedge \ldots \wedge \mathscr{C}$ (p times). Restricting these to the prolonged equation, we get a decreasing filtration

$$\bar{\Omega} = \bar{\mathscr{C}}^0 \supset \bar{\mathscr{C}} = \bar{\mathscr{C}}^1 \supset \bar{\mathscr{C}}^2 \supset \ldots.$$

The Vinogradov's spectral sequence $E_r^{p,q}(R)$ is by definition the spectral sequence associated in the usual way (Godement 1958) with the filtered cochain complex $(\bar{\Omega}, \bar{d})$. By construction, $E_r^{p,q}(R)$ is a first quadrant spectral sequence and converges to $H^*(R_\infty)$, the de Rham cohomology of the manifold R_∞.

The importance of Vinogradov's spectral sequence for our problem lies in the fact that $E_0(R)$ contains the complex (3) as its 0th column and hence the space of conservation laws of R is identified with $E_1^{0,n-1}(R)$.

6. Normal equations

The main theorem of Vinogradov (1978) describing the Vinogradov's spectral sequence of a differential equation is valid for a wide class of not overdetermined equations called in Vinogradov (1984a) normal. Here is a definition of a normal equation $R \subset N_k$ which slightly differs from that of Vinogradov (1984a) but is a bit more suitable in practice: R is *normal* if it enjoys the following two properties:

(1) (cf § 2). *R can be presented globally as $F = 0$ where $F \in \Gamma(\pi_k^* \xi)$, ξ being a vector bundle over M.*

Example. For the BBM equation, we can take $F = u_{xxt} - u_t + uu_x$ (ξ being the trivial one-dimensional bundle).

(2). *For each point $x \in R$, there is, in $T_y^* M$, $y = \pi_k(x)$, a non-characteristic covector for the operator ℓ_F.* For an invariant geometric definition of the 'universal linearisation operator' ℓ_F, see Vinogradov (1984a); here we confine ourselves to its coordinate description (which is sufficient because both bundles π and ξ related to the BBM equation are trivial). The operator ℓ_F is given by the matrix with $m = \dim \pi$ columns and $r = \dim \xi$ rows whose components are

$$(\ell_F)_{ij} = \sum_\sigma \frac{\partial F_i}{\partial u_\sigma^j} D_\sigma, \tag{4}$$

the summation going over all multi-indices σ; $F_1, \ldots, F_r$ being the components of F, and D_σ the total derivative operator ($D_{x_1 \ldots x_n} = D_{x_1} \circ D_{x_2} \circ \ldots \circ D_{x_n}$). A *non-characteristic covector* for the operator ℓ_F at $x \in N_k$ is by definition an element $p = (p_1, \ldots, p_n)$ of $T_y^* M$ ($y = \pi_k(x)$) for which the *symbol* of ℓ_F evaluated at p, i.e., the matrix

$$\left(\sum_{|\sigma| = k} \frac{\partial F_i}{\partial u_\sigma^j}(x) p_\sigma \right)_{1 \leq i \leq r, 1 \leq j \leq m}$$

has rank r, where $|\sigma| = k$ and $p_\sigma = p_{i_1} \ldots p_{i_k}$ for $\sigma = x_{i_1} \ldots x_{i_k}$.

Example. The symbol of ℓ_F for the BBM equation is $p_1^2 p_2$. So for each point of N_3, every covector (p_1, p_2) with $p_1 p_2 \neq 0$ is non-characteristic and condition 2 is satisfied. Note that condition 2 is valid for any equation solvable in the highest derivative. For less trivial examples related to condition 2, see Vinogradov (1984b, § 4.4).

In a similar way, one can easily verify that this class contains almost all 'famous' equations: the KdV, sine-Gordon, wave, Schrödinger equations, Euler equation of hydrodynamics, etc.

We remark that the infinite prolongation R_∞ of a normal equation $R = \{F = 0\}$ is defined by the ideal differentially generated by the components of F, that is, the submanifold $R_\infty \subset N_\infty$ can be described by the equations $D_\sigma F_j = 0$. Note also that $F = 0$ is not overdetermined, i.e., $r \leq m$.

Theorem (*Vinogradov* 1978, 1984*a*). *For a normal equation $R = \{F = 0\}$ all the terms $E_1^{p,q}(R)$ except for those with $p = 0$ or $q = n$ or $q = n - 1$ are zero; $E_1^{1,n-1}(R) \simeq \operatorname{Ker} \bar{\ell}_F^*$.*

Here ℓ_F^* denotes the conjugate operator derived from (4) by the transposition and taking the conjugate of each scalar element of the matrix according to the usual rules: $(K \circ L)^* = L^* \circ K^*$, $D_i^* = -D_i$ and $f^* = f$ for the function coefficient; $\bar{\ell}_F^*$ is the restriction of ℓ_F^* to R_∞.

For a normal equation, R_∞ is an affine bundle over R so that the cohomology of R_∞ coincides with that of R. If the latter is trivial (as is the case with the BBM equation), we conclude from Vinogradov's theorem and the general properties of spectral sequences (Godement 1958) that the space of conservation laws $E_1^{0,n-1}(R)$ is injected by the differential $d_1^{0,n-1}$ into Ker $\bar{\ell}_F^*$ and the image of this injection coincides with Ker $d_1^{1,n-1}$. The computation of Ker $\bar{\ell}_F^*$ constitutes the first and as a rule the most difficult task in finding the conservation laws.

7. Ker $\bar{\ell}_F^*$ for the BBM equation

We come now to realise the plan sketched in §§ 3–6, for the BBM equation. Since $F = u_{xxt} - u_t + uu_x$ we have by (4)

$$\ell_F = D_x^2 D_t - D_t + uD_x + u_x,$$

(u_x standing here for the operator of multiplication by u_x), hence

$$\ell_F^* = -D_x^2 D_t + D_t - uD_x$$

and

$$\bar{\ell}_F^* = -\bar{D}_x^2 \bar{D}_t + \bar{D}_t - u\bar{D}_x.$$

Let $\phi \in \bar{A}$ and $\bar{\ell}_F^*(\phi) = 0$. Choose n so that $\phi \in \bar{A}_n \setminus \bar{A}_{n-1}$ (see § 2). We show first that $n \leqslant 2$ and then $\phi = a + bu + c(u^2 + u_{xt})$ (a, b, c are constants).

First we assume $n \geqslant 2$.

The operator $\bar{\ell}_F^*$ raises the filtration index by two: $\bar{\ell}_F^*(\bar{A}_m) \subset \bar{A}_{m+2}$. Equating the coefficients of u_{n+2}, w_{n+2} and w_{n+1} in $\bar{\ell}_F^*(\phi)$ to zero, we obtain

$$\bar{D}_t\left(\frac{\partial \phi}{\partial u_n}\right) = 0, \qquad \bar{D}_x\left(\frac{\partial \phi}{\partial v_n}\right) = 0, \qquad \bar{D}_x^2 \frac{\partial \phi}{\partial w_n} + 2\bar{D}_x \frac{\partial \phi}{\partial v_{n-1}} = 0.$$

By lemmas 1 and 2, we can write ϕ as

$$\phi = \alpha u_n + \beta v_n + \psi \tag{5}$$

where $\alpha \in C^\infty(x)$, $\beta \in C^\infty(t)$ and $\psi \in \bar{A}_n$ does not depend on u_n, v_n and

$$\bar{D}_x^2\, \partial\psi/\partial w_n + 2\bar{D}_x\, \partial\psi/\partial v_{n-1} = 0. \tag{6}$$

Now substituting (5) into $\partial(\bar{\ell}_F^*\phi)/\partial v_n = 0$, we obtain

$$\bar{D}_x^2\, \partial\psi/\partial v_{n-1} + 2\bar{D}_x\, \partial\psi/\partial w_n = \beta u_1. \tag{7}$$

The equations (6) and (7) can be rewritten by lemma 1 as

$$\bar{D}_x\, \partial\psi/\partial w_n + 2\, \partial\psi/\partial v_{n-1} \in C^\infty(t),$$

$$\bar{D}_x\, \partial\psi/\partial v_{n-1} + 2\, \partial\psi/\partial w_n - \beta u \in C^\infty(t),$$

whence

$$(\bar{D}_x^2 - 4)\, \partial\psi/\partial w_n + 2\beta u \in C^\infty(t).$$

Note that

$$\bar{D}_x^2 - 4 = e^{2x} \circ \bar{D}_x \circ e^{-4x} \circ \bar{D}_x \circ e^{2x}, \tag{8}$$

so

$$\bar{D}_x(e^{-4x}\bar{D}_x(e^{2x}\, \partial\psi/\partial w_n)) + 2e^{-2x}\beta u \in C^\infty(t)\, e^{-2x}. \tag{9}$$

Lemma 3. If $f \in C^\infty(x, t, u)$, then $f \in \operatorname{Im} \bar{D}_x$ if and only if $f = au^2 + b$, with $a, b \in C^\infty(x, t)$ and $a = a_{xx}$.

Proof. Suppose $\bar{D}_x g = f$ with $g \in \bar{A}_m \setminus \bar{A}_{m-1}$. An argument similar to the one used in the proof of lemma 1 shows first that g does not depend on $u_1, \ldots, u_m$ nor on v_m. Then, in the notation of lemma 1, we have

$$Yg = f \qquad \text{and} \qquad Xg = 0.$$

We obtain successively

$$X_1 g = [X, Y]g = Xf = f_u,$$
$$X_2 g = [X, X_1]g = X^2 g = f_{uu},$$
$$X_3 g = [X_1, X_2]g = X_1 f_{uu} - X_2 f_u = 0,$$

since X_1 and X_2 are zero operators on $C^\infty(x, t, u)$. Similarly, we obtain $X_n g = 0$ for $n \geqslant 3$.

If $m \geqslant 3$, then $\partial g/\partial v_{m-1} = -X_m g = 0$. Hence $\partial g/\partial w_m = \partial(\bar{D}_x g)/\partial v_{m-1} = 0$.

If $m = 2$, then by $\partial g/\partial v_1 = -X_2 g$ and $u\, \partial g/\partial w_2 = [Y, X_1]g$, we obtain $g = av_1 + bw_2 + c$ with $a, b, c \in C^\infty(x, t, u)$. It is easy to see that $\bar{D}_x g \in C^\infty(x, t, u)$ if and only if $a \in C^\infty(x, t)$, $b = -a_x$, $a - a_{xx} = 0$, $c = -\frac{1}{2}a_x u^2 + \lambda$ ($\lambda \in C^\infty(x, t)$) and $f = \bar{D}_x g = -\frac{1}{2}au^2 + \lambda_x$.

This lemma shows, by (9), $\beta = 0$. The equation (9) implies now, by lemma 1, that $\partial\psi/\partial w_n \in C^\infty(x, t)$. Hence $\partial\psi/\partial v_{n-1} \in C^\infty(x, t)$.

We have proved the following lemma.

Lemma 4. If $\phi \in \bar{A}_n \setminus \bar{A}_{n-1}$ ($n \geqslant 2$) satisfies $\bar{\ell}^*_F \phi = 0$, then

$$\phi = \alpha u_n + \gamma w_n + \delta v_{n-1} + \chi. \tag{10}$$

Here $\alpha \in C^\infty(x)$, $\gamma, \delta \in C^\infty(x, t)$, $\chi \in \bar{A}_{n-1}$ and

$$\gamma_{xx} + 2\delta_x = 0, \qquad 2\gamma_x + \delta_{xx} = 0, \qquad \partial\chi/\partial v_{n-1} = 0. \tag{11}$$

Suppose now $n \geqslant 3$. Substituting (10) into $\partial(\bar{\ell}^*_F \phi)/\partial u_n = 0$, we obtain

$$\bar{D}_t\, \partial\chi/\partial u_{n-2} = (n+1)\alpha u_1 + \alpha_x u.$$

Lemma 5. If $g \in \bar{A}$ is linear over $C^\infty(x, t)$ in u_i, $i \geqslant 0$, and $g \in \operatorname{Im} \bar{D}_t$, then $g = 0$.

Proof. Suppose $g = \bar{D}_t f$ for some $f \in \bar{A}_m$. An argument similar to the one used in the proof of lemma 2 shows first that f does not depend on $v_1, v_2, \ldots, v_m, w_2, \ldots, w_m$ and

$$\partial f/\partial u_m = X_m f = -[X_0, [X_{m-1}, Y]]f = -X_0 X_{m-1} g = 0,$$

since $X_0 = \partial/\partial u + \partial/\partial u_2 + \ldots, X_{m-1} = \partial/\partial u_{m-1}$ on $\bar{A}_m$ and g is linear in u_i's.

This lemma implies $\alpha = 0$.

Substituting $\phi = \gamma w_n + \delta v_{n-1} + \chi$ in $\bar{\ell}^*_F \phi = 0$ and considering the coefficients of v_{n-1} and w_n, we have

$$\bar{D}_x^2\, \partial\chi/\partial v_{n-2} + 2\bar{D}_x\, \partial\chi/\partial w_{n-1} - \gamma u_2 - (2\gamma_x + \delta)u_1 + \delta_x u = 0 \tag{12}$$

$$2D_x\, \partial\chi/\partial v_{n-2} + \bar{D}_x^2\, \partial\chi/\partial w_{n-1} - 2\gamma u_1 - \gamma_x u = 0. \tag{13}$$

Since the left-hand side of (13) is $\gamma_x u$ modulo $\operatorname{Im} \bar{D}_x$, lemma 3 implies $\gamma_x = 0$ and by (11) $\delta_x = 0$. Rewriting (12) and (13) using lemma 1, we obtain

$$\bar{D}_x\, \partial\chi/\partial v_{n-2} + 2\, \partial\chi/\partial w_{n-1} = \delta u + \gamma u_1 + \lambda, \tag{14}$$

$$2\,\partial\chi/\partial v_{n-2}+\bar{D}_x\,\partial\chi/\partial w_{n-1}=2\gamma u+\mu \tag{15}$$

with λ, $\mu\in C^\infty(t)$. Then, by (12) and (15)

$$(\bar{D}_x^2-4)\,\partial\chi/\partial v_{n-2}=\delta u_1+\gamma u_2-4\gamma u-2\mu. \tag{16}$$

By (8), we obtain $2e^{-2x}\delta u\in\operatorname{Im}\bar{D}_x$, whence by lemma 3,

$$\delta=0. \tag{17}$$

(16) implies then

$$\partial\chi/\partial v_{n-2}=\gamma u+\tfrac{1}{2}\mu+\nu\,e^{2x}+\xi\,e^{-2x} \tag{18}$$

with ν, $\xi\in C^\infty(t)$.

On the other hand, (13) and (14) imply

$$(\bar{D}_x^2-4)\,\partial\chi/\partial w_{n-1}=-2\lambda,$$

whence

$$\partial\chi/\partial w_{n-1}=\tfrac{1}{2}\lambda+\rho\,e^{2x}+\eta\,e^{-2x} \tag{19}$$

with ρ, $\eta\in C^\infty(t)$. By (14), (15), (18) and (19), we obtain $\rho=-\nu$, $\xi=\eta$. Thus we have proved the following.

Lemma 6. If $\phi\in\bar{A}_n\setminus\bar{A}_{n-1}$ $(n\geqslant 3)$ satisfies $\bar{\ell}_F^*\phi=0$, then

$$\phi=\gamma(w_n+uv_{n-2})+\sigma v_{n-2}+\tau w_{n-1}+\omega, \tag{20}$$

where $\gamma\in C^\infty(t)$, $\omega\in\bar{A}_{n-1}$ does not depend on v_{n-2}, v_{n-1}, w_{n-1}, and

$$\sigma=\tfrac{1}{2}\mu+\nu\,e^{2x}+\xi\,e^{-2x} \tag{21}$$

$$\tau=\tfrac{1}{2}\lambda-\nu\,e^{2x}+\xi\,e^{-2x} \tag{22}$$

with λ, μ, ν, $\xi\in C^\infty(t)$.

Now we suppose $n\geqslant 4$. For ϕ of the form (20), we have

$$\frac{\partial(\bar{\ell}_F^*\phi)}{\partial v_{n-2}}=\gamma((2n-3)v_1-(n-2)uu_1)+\gamma_t u+\tau u_2+2\tau_x u_1+\sigma u_1$$

$$-\sigma_x u-\bar{D}_x^2\frac{\partial\omega}{\partial v_{n-3}}-2\bar{D}_x\frac{\partial\omega}{\partial w_{n-2}}=0 \tag{23}$$

$$\frac{\partial(\bar{\ell}_F^*\phi)}{\partial w_{n-1}}=2(n-2)\gamma w_2+2\tau u_1+\tau_x u-2\bar{D}_x\frac{\partial\omega}{\partial v_{n-3}}-\bar{D}_x^2\frac{\partial\omega}{\partial w_{n-2}}=0. \tag{24}$$

From (23), and using $v_1=\bar{D}_x(w_2+\frac{1}{2}u^2)$ and $\tau_{xx}=-2\sigma_x$, we obtain $\gamma_t u\in\operatorname{Im}\bar{D}_x$. Hence by lemma 3, $\gamma_t=0$ and we obtain $\gamma\in\boldsymbol{R}$. On the other hand, (24) implies $\tau_x u\in\operatorname{Im}\bar{D}_x$, whence $\gamma_x=0$. By (22), $\nu=\xi=0$ and then by (21), we have $\sigma_x=0$.

Now (24) and lemma 1 implies

$$2\frac{\partial\omega}{\partial v_{n-3}}+\bar{D}_x\frac{\partial\omega}{\partial w_{n-2}}-2(n-2)\gamma v_1-2\tau u\in C^\infty(t).$$

This implies, together with (8) and (23),

$$\tfrac{1}{3}(n-1)\gamma\,e^{2x}u^2-2\sigma\,e^{2x}u\in\operatorname{Im}\bar{D}_x.$$

Here we used $3\,e^{2x}v_1 \equiv u^2\,e^{2x} (\mathrm{mod\ Im}\,\bar{D}_x)$. Hence by lemma 3, $(\gamma\,e^{2x})_{xx} - \gamma\,e^{2x} = 0$. Thus $\gamma = 0$. This contradicts the definition of n.

Suppose now $n = 3$. Then by lemma 6,

$$\phi = \gamma(w_3 + uv_1) + \sigma v_1 + \tau w_2 + \omega$$

with $\gamma \in C^\infty(t)$, $\omega \in C^\infty(x, t, u, u_1)$ and σ, τ are defined by (21) and (22). From $\partial(\bar{\ell}_F^*\phi)/\partial u_3 = 0$, we obtain $\bar{D}_t\,\partial\omega/\partial u_1 = 0$, whence $\omega = \varepsilon u_1 + \omega'$ with $\varepsilon \in C^\infty(x)$ and $\omega' \in \bar{A}_0$. Then $\partial^2 \bar{\ell}_F^*\phi/\partial v_1^2 = 6\gamma$, whence $\gamma = 0$.

Finally suppose $n \leqslant 2$. Then lemma 4 and the equations $\partial(\bar{\ell}_F^*\phi)/\partial u_3 = \partial(\bar{\ell}_F^*\phi)/\partial u_2 = 0$ imply

$$\phi = \gamma(w_2 + \tfrac{1}{2}u^2) + \varepsilon u_1 + \delta v_1 + \zeta u + \theta$$

with $\varepsilon, \gamma, \delta, \zeta, \theta \in C^\infty(x, t)$ satisfying $2\gamma_x + \delta_{xx} = \zeta_t = 0$.

Then equating the coefficients of $u_1 v_1$, u_1^2, w_2 and u^2 in $\bar{\ell}_F^*\phi$ to zero, it follows $\delta = \varepsilon = 0$ and $\zeta, \gamma \in \boldsymbol{R}$. Then $\bar{\ell}_F^*\phi = \bar{\ell}_F^*\theta = 0$ implies $\theta \in \boldsymbol{R}$. Thus we have proved that $\mathrm{Ker}\,\bar{\ell}_F^*$ is three dimensional and generated by 1, u, $u^2 + \frac{1}{2}u_{xt}$.

The general properties of Vinogradov's spectral sequence (§ 6) lead one immediately to the following.

Theorem. The dimension of the space of conservation laws of the BBM equation is not greater than 3.

Since we already know three independent conservation laws of the BBM equation, there is no need to pick out those elements of $E_1^{1,n-1}(R)$ which are in $\mathrm{Ker}\,d_1^{1,n-1}$ and to inverse them (i.e. find corresponding classes in $\bar{H}^{n-1}(R)$). These procedures, however, might prove useful for other equations and are described in Duzhin (1982), Tsujishita (1982) and Vinogradov (1984 a, b).

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